Acyclic Edge Coloring of Triangle Free Planar Graphs

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Abstract

An *acyclic* edge coloring of a graph is a proper edge coloring such that there are no bichromatic cycles. The *acyclic chromatic index* of a graph is the minimum number k such that there is an acyclic edge coloring using k colors and is denoted by $a'(G)$. It was conjectured by Alon, Sudakov and Zaks (and much earlier by Fiamcik) that $a'(G) \leq \Delta + 2$, where $\Delta = \Delta(G)$ denotes the maximum degree of the graph.

If every induced subgraph H of G satisfies the condition $|E(H)| \leq 2|V(H)| - 1$, we say that the graph G satisfies *Property A*. In this paper, we prove that if G satisfies *Property A*, then $a'(G) \leq \Delta + 3$. Triangle free planar graphs satisfy *Property A*. We infer that $a'(G) \leq \Delta + 3$, if G is a triangle free planar graph. Another class of graph which satisfies *Property A* is 2-fold graphs (union of two forests).

Keywords: Acyclic edge coloring, acyclic edge chromatic number, planar graphs.

1   Introduction

All graphs considered in this paper are finite and simple. A proper *edge coloring* of $G = (V, E)$ is a map $c : E \to C$ (where C is the set of available *colors*) with $c(e) \neq c(f)$ for any adjacent edges e, f. The minimum number of colors needed to properly color the edges of G, is called the chromatic index of G and is denoted by $\chi'(G)$. A proper edge coloring c is called acyclic if there are no bichromatic cycles in the graph. In other words an edge coloring is acyclic if the union of any two color classes induces a set of paths (i.e., linear forest) in G. The *acyclic edge chromatic number* (also called *acyclic chromatic index*), denoted by $a'(G)$, is the minimum number of colors required to acyclically edge color G. The concept of *acyclic coloring* of a graph was introduced by Grünbaum [18]. The *acyclic chromatic index* and its vertex analogue can be used to bound other parameters like *oriented chromatic number* and *star chromatic number* of a graph, both of which have many practical applications, for example, in wavelength routing in optical networks ([4], [20]). Let $\Delta = \Delta(G)$ denote the maximum degree of a vertex in graph G. By Vizing's theorem, we have $\Delta \leq \chi'(G) \leq \Delta + 1$(see [10] for proof). Since any acyclic edge coloring is also proper, we have $a'(G) \geq \chi'(G) \geq \Delta$.

It has been conjectured by Alon, Sudakov and Zaks [2] (and much earlier by Fiamcik [11]) that $a'(G) \leq \Delta + 2$ for any G. Using probabilistic arguments Alon, McDiarmid and Reed [1] proved that $a'(G) \leq 60\Delta$. The best known result up to now for arbitrary graph, is by Molloy and Reed [21] who showed that $a'(G) \leq 16\Delta$. Muthu, Narayanan and Subramanian [22] proved that $a'(G) \leq 4.52\Delta$ for graphs G of girth at least 220 (*Girth* is the length of a shortest cycle in a graph).

Though the best known upper bound for general case is far from the conjectured $\Delta + 2$, the conjecture has been shown to be true for some special classes of graphs. Alon, Sudakov and Zaks [2] proved that there exists a constant k such that $a'(G) \leq \Delta + 2$ for any graph G whose girth is at least $k\Delta \log \Delta$. They also proved that $a'(G) \leq \Delta + 2$ for almost all Δ-regular graphs. This result was improved by Nešetřil and Wormald [24] who showed that for a random Δ-regular graph $a'(G) \leq \Delta + 1$. Muthu, Narayanan and Subramanian proved the conjecture for grid-like graphs [23]. In fact they gave a better bound of $\Delta + 1$ for these class of graphs. From Burnstein's [9] result it follows that the conjecture is true for subcubic graphs. Skulrattankulchai [26] gave a polynomial time algorithm to color a subcubic graph using $\Delta + 2 = 5$ colors. Fiamcik [13], [12] proved that every subcubic graph, except for K_4 and $K_{3,3}$, is acyclically edge colorable using 4 colors.

Determining $a'(G)$ is a hard problem both from theoretical and algorithmic points of view. Even for the simple and highly structured class of complete graphs, the value of $a'(G)$ is still not determined exactly. It has also been shown by

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Alon and Zaks [3] that determining whether $a'(G) \le 3$ is NP-complete for an arbitrary graph G. The vertex version of this problem has also been extensively studied (see [18], [9], [8]). A generalization of the acyclic edge chromatic number has been studied: The *r-acyclic edge chromatic number* $a'_r(G)$ is the minimum number of colors required to color the edges of the graph G such that every cycle C of G has at least $\min\{|C|, r\}$ colors (see [16], [17]).

Our Result: The acyclic chromatic index of planar graphs has been studied previously. Fiedorowicz, Hauszczak and Narayanan [15] gave an upper bound of $2\Delta + 29$ for planar graphs. Independently Hou, Wu, GuiZhen Liu and Bin Liu [19] gave an upper bound of $max(2\Delta - 2, \Delta + 22)$. Note that for $\Delta \ge 24$, it is equal to $2\Delta - 2$. Basavaraju and Chandran [7] improved the bound significantly to $\Delta + 12$.

The acyclic chromatic index of special classes of planar graphs characterized by some lower bounds on girth or the absence of short cycles have also been studied. In [19] an upper bound of $\Delta + 2$ for planar graphs of girth at least 5 has been proved. Fiedorowicz and Borowiecki [14] proved an upper bound of $\Delta + 1$ for planar graphs of girth at least 6 and an upper bound of $\Delta + 15$ for planar graphs without cycles of length 4. In [15], an upper bound of $\Delta + 6$ for triangle free planar graphs has been proved. In this paper we improve the bound to $\Delta + 3$. In fact we prove a more general theorem as described below:

Property A : Let G be a simple graph. If every induced subgraph H of G satisfies the condition $|E(H)| \le 2|V(H)| - 1$, we say that the graph G satisfies *Property A*. If G satisfies *Property A*, then every subgraph of G also satisfies *Property A*.

In this paper, we prove the following theorem:

Theorem 1. *If a graph G satisfies Property A, then $a'(G) \le \Delta(G) + 3$.*

Note that triangle free planar graphs, 2-degenerate graphs, 2-fold graphs (union of two forests), etc. are some classes of graphs which satisfy *Property A*. The following corollary is obvious.

Corollary 1. *If G is a triangle free planar graph, then $a'(G) \le \Delta + 3$.*

Note that this is the best result known for triangle free planar graphs and 2-fold graphs. The earlier known bound for these classes of graphs was $\Delta + 6$ by [15]. In case of 2-degenerate graphs a tight bound of $\Delta + 1$ has been proved in [5].

Our proof is constructive and yields an efficient polynomial time algorithm. We have presented the proof in a non-algorithmic way. But it is easy to extract the underlying algorithm from it.

2 Preliminaries

Let $G = (V, E)$ be a simple, finite and connected graph of n vertices and m edges. Let $x \in V$. Then $N_G(x)$ will denote the neighbours of x in G. For an edge $e \in E$, $G - e$ will denote the graph obtained by deletion of the edge e. For $x, y \in V$, when $e = (x, y) = xy$, we may use $G - \{xy\}$ instead of $G - e$. Let $c : E \to \{1, 2, \ldots, k\}$ be an *acyclic edge coloring* of G. For an edge $e \in E$, $c(e)$ will denote the color given to e with respect to the coloring c. For $x, y \in V$, when $e = (x, y) = xy$ we may use $c(x, y)$ instead of $c(e)$. For $S \subseteq V$, we denote the induced subgraph on S by $G[S]$.

Many of the definitions, facts and lemmas that we develop in this section are already present in our earlier papers [6], [5]. We include them here for the sake of completeness. The proofs of the lemmas will be omitted whenever it is available in [6], [5].

Partial Coloring: Let H be a subgraph of G. Then an edge coloring c' of H is also a partial coloring of G. Note that H can be G itself. Thus a coloring c of G itself can be considered a partial coloring. A partial coloring c of G is said to be a proper partial coloring if c is proper. A proper partial coloring c is called acyclic if there are no bichromatic cycles in the graph. Sometimes we also use the word valid coloring instead of acyclic coloring. Note that with respect to a partial coloring c, $c(e)$ may not be defined for an edge e. So, whenever we use $c(e)$, we are considering an edge e for which $c(e)$ is defined, though we may not always explicitly mention it.

Let c be a partial coloring of G. We denote the set of colors in the partial coloring c by $C = \{1, 2, \ldots, k\}$. For any vertex $u \in V(G)$, we define $F_u(c) = \{c(u, z) | z \in N_G(u)\}$. For an edge $ab \in E$, we define $S_{ab}(c) = F_b - \{c(a, b)\}$. Note that $S_{ab}(c)$ need not be the same as $S_{ba}(c)$. We will abbreviate the notation to F_u and S_{ab} when the coloring c is understood from the context.

To prove the main result, we plan to use contradiction. Let G be the minimum counter example with respect to the number of edges for the statement in the theorems that we plan to prove. Let $G = (V, E)$ be a graph on m edges where

$m \geq 1$. We will remove an edge $e = (x, y)$ from G and get a graph $G' = (V, E')$. By the minimality of G, the graph G' will have an acyclic edge coloring $c : E' \to \{1, 2, \ldots, t\}$, where t is the claimed upper bound for $a'(G)$. Our intention will be to extend the coloring c of G' to G by assigning an appropriate color for the edge e thereby contradicting the assumption that G is a minimum counter example.

The following definitions arise out of our attempt to understand what may prevent us from extending a partial coloring of $G - e$ to G.

Maximal bichromatic Path: An (α, β)-maximal bichromatic path with respect to a partial coloring c of G is a maximal path consisting of edges that are colored using the colors α and β alternatingly. An (α, β, a, b)-maximal bichromatic path is an (α, β)-maximal bichromatic path which starts at the vertex a with an edge colored α and ends at b. We emphasize that the edge of the (α, β, a, b)-maximal bichromatic path incident on vertex a is colored α and the edge incident on vertex b can be colored either α or β. Thus the notations (α, β, a, b) and (α, β, b, a) have different meanings. Also note that any maximal bichromatic path will have at least two edges. The following fact is obvious from the definition of proper edge coloring:

Fact 1. *Given a pair of colors α and β of a proper coloring c of G, there can be at most one maximal (α, β)-bichromatic path containing a particular vertex v, with respect to c.*

A color $\alpha \neq c(e)$ is a *candidate* for an edge e in G with respect to a partial coloring c of G if none of the adjacent edges of e are colored α. A candidate color α is *valid* for an edge e if assigning the color α to e does not result in any bichromatic cycle in G.

Let $e = (a, b)$ be an edge in G. Note that any color $\beta \notin F_a \cup F_b$ is a candidate color for the edge ab in G with respect to the partial coloring c of G. A sufficient condition for a candidate color being valid is captured in the Lemma below (See Appendix for proof):

Lemma 1. *[6] A candidate color for an edge $e = ab$, is valid if $(F_a \cap F_b) - \{c(a, b)\} = (S_{ab} \cap S_{ba}) = \emptyset$.*

Now even if $S_{ab} \cap S_{ba} \neq \emptyset$, a candidate color β may be valid. But if β is not valid, then what may be the reason? It is clear that color β is not *valid* if and only if there exists $\alpha \neq \beta$ such that a (α, β)-bichromatic cycle gets formed if we assign color β to the edge e. In other words, if and only if, with respect to coloring c of G there existed a (α, β, a, b) maximal bichromatic path with α being the color given to the first and last edge of this path. Such paths play an important role in our proofs. We call them *critical paths*. It is formally defined below:

Critical Path: Let $ab \in E$ and c be a partial coloring of G. Then a (α, β, a, b) maximal bichromatic path which starts out from the vertex a via an edge colored α and ends at the vertex b via an edge colored α is called an (α, β, ab) critical path. Note that any critical path will be of odd length. Moreover the smallest length possible is three.

An obvious strategy to extend a valid partial coloring c of G would be to try to assign one of the candidate colors to an uncolored edge e. The condition that a candidate color being not valid for the edge e is captured in the following fact.

Fact 2. *Let c be a partial coloring of G. A candidate color β is not valid for the edge $e = (a, b)$ if and only if $\exists \alpha \in S_{ab} \cap S_{ba}$ such that there is a (α, β, ab) critical path in G with respect to the coloring c.*

Actively Present: Let c be a partial coloring of G. Let $a \in N_G(x)$ and let $c(x, a) = \alpha$. Let $\beta \in S_{xa}$. Color β is said to be *actively present* in a set S_{xa} with respect to the edge xy, if there exists a (α, β, xy) critical path. When the edge xy is understood in the context, we just say that β is actively present in S_{xa}.

Color Exchange: Let c be a partial coloring of G. Let $u, i, j \in V(G)$ and $ui, uj \in E(G)$. We define *Color Exchange* with respect to the edge ui and uj, as the modification of the current partial coloring c by exchanging the colors of the edges ui and uj to get a partial coloring c', i.e., $c'(u, i) = c(u, j)$, $c'(u, j) = c(u, i)$ and $c'(e) = c(e)$ for all other edges e in G. The color exchange with respect to the edges ui and uj is said to be proper if the coloring obtained after the exchange is proper. The color exchange with respect to the edges ui and uj is *valid* if and only if the coloring obtained after the exchange is acyclic. The following fact is obvious:

Fact 3. *Let c' be the partial coloring obtained from a valid partial coloring c by the color exchange with respect to the edges ui and uj. Then the partial coloring c' will be proper if and only if $c(u, i) \notin S_{uj}$ and $c(u, j) \notin S_{ui}$.*

The color exchange is useful in breaking some critical paths as is clear from the following lemma (See Appendix for proof):

Lemma 2. *[6], [5] Let $u, i, j, a, b \in V(G)$, $ui, uj, ab \in E$. Also let $\{\lambda, \xi\} \in C$ such that $\{\lambda, \xi\} \cap \{c(u, i), c(u, j)\} \neq \emptyset$ and $\{i, j\} \cap \{a, b\} = \emptyset$. Suppose there exists an (λ,ξ,ab)-critical path that contains vertex u, with respect to a valid partial coloring c of G. Let c' be the partial coloring obtained from c by the color exchange with respect to the edges ui and uj. If c' is proper, then there will not be any (λ,ξ,ab)-critical path in G with respect to the partial coloring c'.*

The following is the main result of [6]. We will need this result for proving our theorems.

Lemma 3. *[6] Let G be a connected graph on n vertices, $m \leq 2n - 1$ edges and maximum degree $\Delta \leq 4$, then $a'(G) \leq 6$.*

3   Proof of Theorem 1

Proof. A well-known strategy that is used in proving coloring theorems in the context of sparse graphs is to make use of induction combined with the fact that there are some *unavoidable* configurations in any such graphs. Typically the existence of these *unavoidable* configurations are proved using the so called *charging and discharging argument* (See [25], for a comprehensive exposition). $Lemma$ 4 will establish that one of the five configurations $B1, \ldots, B5$ is unavoidable in any graph G that satisfies $Property$ A. Loosely speaking, for the purpose of this paper, a *configuration* is a subset Q of V, where one special vertex $v \in Q$ is called the *pivot* of the configuration and $Q = \{v\} \cup N(v)$. Besides v, one more vertex in Q will be given a special status: This vertex, called the *co-pivot* of the configuration, is selected such that it is a vertex of smallest degree in $N(v)$ and will be denoted by u. Moreover the vertices of $N(v)$ will be partitioned into two sets namely $N'(v)$ and $N''(v)$. The members of $N'(v)$ and $N''(v)$ are explicitly defined for each configuration.

Lemma 4. *Let G be a simple graph such that $|E(G)| \leq 2|V(G)| - 1$ with minimum degree $\delta \geq 2$. Then there exists a vertex v in G with $k = deg(v)$ neighbours such that at least one of the following is true:*

(B1) $k = 2$,

(B2) $k = 3$ with $N(v) = \{u, v_1, a\}$ such that $deg(u), deg(v_1) \leq 4$. $N'(v) = \{u, v_1\}$ and $N''(v) = \{a\}$,

(B3) $k = 5$ with $N(v) = \{u, v_1, v_2, a, b\}$ such that $deg(u), deg(v_1), deg(v_2) \leq 3$. $N'(v) = \{u, v_1, v_2\}$ and $N''(v) = \{a, b\}$,

(B4) $k = 6$ with $N(v) = \{u, v_1, v_2, v_3, v_4, a\}$ such that $deg(u), deg(v_1), deg(v_2), deg(v_3), deg(v_4) \leq 3$. $N'(v) = \{u, v_1, v_2, v_3, v_4\}$ and $N''(v) = \{a\}$,

(B5) $k \geq 7$ with $N(v) = \{u, v_1, v_2, \ldots, v_{k-1}\}$ such that $deg(u), deg(v_1), deg(v_2), \ldots, deg(v_{k-1}) \leq 3$. $N'(v) = \{u, v_1, v_2, \ldots, v_{k-1}\}$.

Proof. We use the discharging method to prove the lemma. Let $G = (V, E)$, $\delta \geq 2$, $|V| = n$ and $|E| = m \leq 2n - 1$. We define a mapping $\phi : V \longmapsto \mathbb{R}$ using the rule $\phi(v) = deg(v) - 4$ for each $v \in V$. The value $\phi(v)$ is called the charge on the vertex v. Since $m \leq 2n - 1$, it is easy to see that $\sum_{v \in V} \phi(v) \leq -2$. Now we redistribute the charges on the vertices using the following rule. (This procedure is usually known as *discharging*: Note that the total charge has to remain same after the discharging.)

- If vertex v has degree at least 5, then it gives a charge of $\frac{1}{2}$ to each of its 3-degree neighbours.

After $discharging$, each vertex v has a new charge $\phi'(v)$. Now since the total charge is conserved, we have $\sum_{v \in V} \phi(v) = \sum_{v \in V} \phi'(v) \leq -2$. Now suppose the graph G has none of the configurations $B1, \ldots, B5$. Then we will show that for each vertex v of G, $\phi'(v) \geq 0$ and therefore $\sum_{v \in V} \phi'(v) \geq 0$, a contradiction. Since G does not have configuration $B1$, we have $\delta \geq 3$. Now we calculate the charge on each vertex v of G as follows:

- If $deg(v) = 3$: Since G does not have configuration $B2$, at least two of the neighbours have degree at least 5. Thus v receives a charge of $\frac{1}{2}$ each from at least two of its neighbours. Thus $\phi'(v) \geq deg(v) - 4 + 2 \cdot \frac{1}{2} = 0$.

- If $deg(v) = 4$: A four degree vertex does not give or receive any charge. Thus $\phi'(v) = \phi(v) = deg(v) - 4 = 0$.

- If $deg(v) = 5$: Since G does not have configuration $B3$, at most two of the neighbours have degree 3. Thus v gives a charge of $\frac{1}{2}$ each to at most two of its neighbours. Thus $\phi'(v) \geq deg(v) - 4 - 2 \cdot \frac{1}{2} = 0$.

- If $deg(v) = 6$: Since G does not have configuration $B4$, at most four of the neighbours have degree 3. Thus v gives a charge of $\frac{1}{2}$ each to at most four of its neighbours. Thus $\phi'(v) \geq deg(v) - 4 - 4 \cdot \frac{1}{2} = 0$.

- If $deg(v) \geq 7$: Since G does not have configuration $B5$, at most $deg(v) - 1$ of the neighbours have degree 3. Thus v gives a charge of $\frac{1}{2}$ each to at most $deg(v) - 1$ of its neighbours. Thus $\phi'(v) \geq deg(v) - 4 - (deg(v) - 1) \cdot \frac{1}{2} = \frac{1}{2}(deg(v) - 7) \geq 0$.

Thus we have established that $\phi'(v) \geq 0$, $\forall v \in V$ and therefore $\sum_{v \in V} \phi'(v) \geq 0$, a contradiction. $\qquad\square$

We prove the theorem by way of contradiction. Let G be a minimum counter example (with respect to the number of edges) for the theorem statement among the graphs satisfying $Property\ A$. Clearly G is 2-*connected* since if there are cut vertices in G, the acyclic edge coloring of the blocks $G_1, G_2, \ldots, G_k$ of G can easily be extended to G (Note that each block satisfies the $Property\ A$ since they are subgraphs of G). Thus we have, $\delta(G) \geq 2$. Also from $Lemma\ 3$, we know that $a'(G) \leq \Delta + 3$, when $\Delta \leq 4$. Therefore we can assume that $\Delta \geq 5$. Thus we have,

Assumption 1. *For the minimum counter example G, $\delta(G) \geq 2$ and $\Delta(G) \geq 5$.*

By $Lemma\ 4$, graph G has a vertex v, such that it is the pivot of one of the configurations $B1, \ldots, B5$. We present the proof in two parts based on the configuration that v belongs to. The first part deals with the case when G has a vertex v that belongs to configuration $B2$, $B3$, $B4$ or $B5$ and the second part deals with the case when G does not have a vertex v that belongs to configuration $B2$, $B3$, $B4$ or $B5$.

3.1 There exists a vertex v that belongs to configuration $B2$, $B3$, $B4$ or $B5$

Let v be a vertex such that it is the pivot of one of the configurations $B2, \ldots, B5$ and let u be the co-pivot. Since G is a minimum counter example, the graph $G - \{vu\}$ is acyclically edge colorable using $\Delta + 3$ colors. Let c' be a valid coloring of $G - \{vu\}$ and hence a partial coloring of G. We now try to extend c' to a valid coloring of G. With respect to the partial coloring c' let $F'_v(c') = \{c'(v, x) | x \in N'(v)\}$ and $F''_v(c') = \{c'(v, x) | x \in N''(v)\}$ i.e., $F''_v = F_v - F'_v$.

Claim 1. *With respect to any valid coloring c' of $G - \{uv\}$, $|F_u \cap F_v| \geq 1$*

Proof. Suppose not. Then $S_{vu} \cap S_{uv} = \emptyset$ and by $Lemma\ 1$, all the candidate colors are valid for the edge vu. It is easy to verify that irrespective of which configuration v belongs to, $|F_u \cup F_v| \leq \Delta - 1 + 2 = \Delta + 1$. Therefore there are at least two candidate colors for the edge vu which are also valid, a contradiction to the assumption that G is a counter example. $\qquad\square$

Claim 2. *$\forall x \in N(v)$, we have $deg(x) \geq 3$.*

Proof. Suppose not. Then by $Assumption\ 1$, it is clear that the degree of the co-pivot, $deg(u) = 2$. Let $N(u) = \{v, v'\}$. It is easy to verify from the description of configurations $B2 - B5$ and the fact that $deg(u) = 2$ that there can be at most two vertices in $N(v)$ whose degrees are greater than 3. By Claim 1, we know that $c'(u, v') \in F_v$. Let $D_v = D_v(c') = \{c'(v, x) | deg_G(x) \leq 3\}$. Clearly have $|D_v| \leq 2$.

If $c'(u, v') \in F_v - D_v$, then let $c = c'$. Else if $c'(u, v') \in D_v$, then recolor edge uv' using a color from $C - (S_{uv'} \cup D_v)$ to get a coloring c (Note that $|C - (S_{uv'} \cup D_v)| \geq \Delta + 3 - (\Delta - 1 + 2) = 2$ and since u' is a pendant vertex in $G - \{uu'\}$ the recoloring is valid). Now if $c(u, v') \notin F_v$, then it a contradiction to Claim 1. Thus $c(u, v') \in F_v - D_v$.

With respect to coloring c, let $c(u, v') = c(v, v_1)$. Now there are at least four candidate colors for the edge uv since $|F_u \cup F_v| \leq \Delta - 1$. If none of them are valid then they all have to be actively present in S_{vv_1}, implying that $|S_{vv_1}| \geq 4$, a contradiction since $|S_{vv_1}| \leq 3$. Thus there exists a color valid for the edge uv, a contradiction to the assumption that G is a counter example. $\qquad\square$

Claim 3. *$deg(v) > 3$. Therefore v does not belong to $Configuration\ B2$.*

Proof. Suppose v belongs to $Configuration\ B2$. Let $N(v) = \{u, v_1, a\}$ such that $deg(u) \leq 4$ and $deg(v_1) \leq 4$. We also know from $Claim\ 2$ that $deg(u) \geq 3$. Let $N(u) = \{x, y, v\}$, if $deg(u) = 3$ and let $N(u) = \{x, y, z, v\}$, if $deg(u) = 4$. Now the following cases occur:

- $|F_u \cap F_v| = 2$.

  Let $F_u \cap F_v = \{1, 2\}$. Also let $c(u, x) = c(v, a) = 1$ and $c(u, y) = c(v, v_1) = 2$. Since $|F_v \cup F_u| \leq 3$, there are at least Δ candidate colors for the edge vu. If none of them are valid then all those colors are actively present either in S_{vv_1} or S_{va}. Recalling that $|S_{va}| \leq \Delta - 1$ we can infer that there is at least one color $\alpha \in C - (F_v \cup F_u)$ that does not belong to S_{va}. Note that $|S_{vv_1} \cup F_v \cup F_u| \leq 6$ since $|S_{vv_1}| \leq 3$ and $|F_v \cup F_u| \leq 3$. Since $\Delta \geq 5$, we have $C - (S_{vv_1} \cup F_v \cup F_u) \neq \emptyset$. Recolor the edge vv_1 with the a color β from $C - (S_{vv_1} \cup F_v \cup F_u)$ to get a coloring c. The coloring c is valid because if a bichromatic cycle gets created due to recoloring then it has to be a $(\beta, 1)$ bichromatic cycle since $c(v, a) = 1$, implying that there existed a $(1, \beta, vv_1)$ critical path with respect to coloring c'. Recall that color β was not valid for the edge vu. Since $\beta \notin S_{vv_1}$, it implies that color β was actively present in S_{va}. This implies that there existed a $(1, \beta, vu)$ critical path with respect to coloring c'. Therefore by $Fact$ 1, there cannot exists a $(1, \beta, vv_1)$ critical path with respect to c', a contradiction. Thus the coloring c is valid. Now in c we have $F_v \cap F_u = \{1\}$ and $\alpha \notin S_{va}$. Thus color α is valid for the edge vu, a contradiction to the assumption that G is a counter example.

- $|F_u \cap F_v| = 1$.

  Let $F_u \cap F_v = \{1\}$. Now if $c'(v, v_1) \in F_u \cap F_v$, then let $c'' = c'$. Otherwise let $c(u, x) = c(v, a) = 1$ and $c'(v, v_1) = 4$. If $deg(u) \leq 3$, then $|F_v \cup F_u| = 3$. Now there are at least Δ candidate colors for the edge vu. If none of them are valid then all the candidate colors are actively present in S_{va}, a contradiction since $|S_{va}| \leq \Delta - 1$. Thus there exists a valid color for the edge vu. Thus $deg(u) = 4$ and $|F_v \cup F_u| = 4$. Let $c(u, y) = 2$ and $c(u, z) = 3$. There are at least $\Delta - 1$ candidate colors for the edge vu. If none of them are valid then all the candidate colors are actively present in S_{va} and S_{ux}, implying that $S_{va} = S_{ux} = C - \{1, 2, 3, 4\}$. Now recolor edge ux using color 4 to get a coloring c''. It is valid by $Lemma\ 1$ since $S_{ux} \cap S_{xu} = \emptyset$ (Note that $S_{xu}(c') = \{2, 3\}$).

  In both cases we have $\{c''(v, v_1)\} = F_u \cap F_v$. If none of the colors are valid for the edge vu, then all the candidate colors are actively present in $S_{vv'}$, implying that $S_{vv_1} = C - \{1, 2, 3, 4\}$. Since $\Delta \geq 5$, we have $|C - \{1, 2, 3, 4\}| \geq 8 - 4 = 4$. But $|S_{vv_1}| \leq 3$, a contradiction. Thus there exists a color valid for the edge vu, a contradiction to the assumption that G is a counter example. $\qquad\qquad\square$

In view of $Claim\ 3$ we have $deg(v) > 3$. Therefore v belongs to configurations $B3$, $B4$ or $B5$. Now in view of $Claim\ 2$, we have the following observation:

Observation 1. $deg(u) = 3$. *Let $N(u) = \{v, w, z\}$.*

In view of $Claim\ 1$, we have the following two cases:

3.1.1   case 1: $|F_v \cap F_u| = 2$

Note that in this case $F_u \subseteq F_v$. Let $F_u = F_u \cap F_v = \{1, 2\}$. Let $c'(u, z) = 1$ and $c'(u, w) = 2$.

Claim 4. $F_u \nsubseteq F_v'$. *Therefore $F_v'' \cap F_u \neq \emptyset$.*

Proof. Suppose not. Then let $c'(v, v_1) = c'(u, z) = 1$ and $c'(v, v_2) = c'(u, w) = 2$ (See the statement of $Lemma\ 4$ for the naming convention of the neighbours of v). Since $|F_u \cup F_v| \leq \Delta - 1$, there are at least four candidate colors for the edge vu. If none of the candidate colors are valid for the edge vu, then we should have $S_{vv_1} \subset C - (F_u \cup F_v)$ and $S_{vv_2} \subset C - (F_u \cup F_v)$ since $|S_{vv_1}| = 2$ and $|S_{vv_2}| = 2$. Also $S_{vv_1} \cap S_{vv_2} = \emptyset$. Note that $C - (S_{vv_1} \cup F_v \cup F_u) \neq \emptyset$ since $|F_u \cup F_v| \leq \Delta - 1$ and $|S_{vv_1}| = 2$. Now assign a color from $C - (S_{vv_1} \cup F_u \cup F_v)$ to the edge vv_1 to get a coloring c. Recall that $S_{vv_1} \subset C - (F_u \cup F_v)$ and therefore $S_{vv_1} \cap S_{v_1v} = \emptyset$. Thus by $Lemma\ 1$, the coloring c is valid. With respect to the coloring c, $F_u \cap F_v = \{2\}$ and therefore if a candidate color is not valid for the edge vu, it has to be actively present in S_{vv_2}. Let $\alpha \in S_{vv_1}$. Clearly $\alpha \in C - (F_u \cup F_v)$ is a candidate color for the edge vu. Now since $\alpha \notin S_{vv_2}$ (recall that $S_{vv_1} \cap S_{vv_2} = \emptyset$), color α is valid for the edge vu, a contradiction to the assumption that G is a counter example. $\qquad\square$

In view of $Claim$ 4, $F_v'' \cap F_u \neq \emptyset$ and therefore $F_v'' \neq \emptyset$. It follows that vertex v does not belong to configuration $B5$. Recalling $Claim$ 3, we infer that the vertex v belongs to either configuration $B3$ or $B4$. We take care of these two configurations separately below:

subcase 1.1: v belongs to configuration $B3$.
Since $deg(v) = 5$, we have $|F_v| = 4$. Let $F_v = \{1, 2, 3, 4\}$. Recall that by Claim 4, we have $F_v'' \cap F_u \neq \emptyset$. Without loss of generality let $c'(u, z) = c'(v, a) = 1$ and $c'(u, w) = 2$. Now there are $\Delta - 1$ candidate colors for the edge vu. If none of them are valid then all these candidate colors are actively present in at least one of S_{uz} and S_{uw}. Let $Y = C - \{1, 2, 3, 4\}$. We make the following claim:

Claim 5. *With respect to any valid coloring c' of $G - \{uv\}$, $Y = S_{uz}$ and $Y = S_{uw}$.*

Proof. We use contradiction to prove the claim. Firstly we make the following subclaim:

subclaim 5.1: *With respect to any valid coloring c' of $G - \{uv\}$, one of S_{uz} or S_{uw} is Y.*

Proof. Suppose not. Then $Y \neq S_{uz}$ and $Y \neq S_{uw}$. Note that $|Y| = \Delta - 1$ while $|S_{uz}| \leq \Delta - 1$ and $|S_{uw}| \leq \Delta - 1$. Therefore there exist colors $\alpha, \beta \in Y$ such that $\alpha \notin S_{uz}$ and $\beta \notin S_{uw}$. Note that $\alpha \neq \beta$ since otherwise color $\alpha = \beta$ will be valid for the edge vu as there cannot exist a $(1, \alpha, vu)$ or $(2, \alpha, vu)$ critical path with respect to c'. It follows that α is actively present in S_{uw} and β is actively present in S_{uz}. Hence there exist $(2, \alpha, vu)$ and $(1, \beta, vu)$ critical paths. Now recolor edge uz using color α to get a coloring c''. The recoloring is valid since if there is a bichromatic cycle then it has to be a $(\alpha, 2)$ bichromatic cycle, implying that there existed a $(2, \alpha, uz)$ critical path in c', a contradiction in view of Fact 1 as there already existed a $(2, \alpha, vu)$ critical path. With respect to coloring c'', $F_v \cap F_u = \{2\}$ and therefore if a candidate color is not valid for the edge vu, it has to be actively present in S_{uw}. Now color $\beta \notin S_{uw}$ and hence color β is valid for the edge vu, a contradiction to the assumption that G is a counter example. □

With respect to any valid coloring c' of $G - \{uv\}$, in view of $subclaim$ 5.1, let $u' \in \{w, z\}$ be such that $S_{uu'} = Y$. Let $\{u''\} = \{w, z\} - \{u'\}$. Now for contradiction assume that $S_{uu''} \neq Y$. Then clearly there exists a color $\alpha \in Y$ such that $\alpha \notin S_{uu''}$.

subclaim 5.2: *With respect to any valid coloring c' of $G - \{uv\}$, if exactly one of S_{uw} and S_{uz} is Y, say $S_{uu'} = Y$, then all the colors of Y are actively present in $S_{uu'}$ and $c'(u, u') \in F_v''$.*

Proof. Recolor the edge uu'' with the color α to get a coloring c''. Since $\alpha \notin S_{uu''}$ and α is not valid for the edge vu, color α is actively present in $S_{uu'}$ i.e., with respect to coloring c', there exists a (γ, α, vu) critical path, where $\gamma = c'(u, u')$. Thus by $Fact$ **??**, there cannot exist a (γ, α, uu'') critical path and hence the coloring c'' is valid for the edge uu''. With respect to coloring c'', $F_v \cap F_u = \{2\}$. Now all the $\Delta - 2$ colors from $Y - \{\alpha\}$ are candidates for the edge vu. If any one of them is valid we are done. Thus none of them are valid and hence they all have to be actively present in $S_{uu'}$. Recalling that the color α was actively present in $S_{uu'}$ we infer that all the colors of Y are in fact actively present in $S_{uu'}$.

Now these colors will also be actively present in $S_{vv'}$, where $v' \in N(v)$ is such that $c'(v, v') = c'(u, u')$. This implies that $|S_{vv'}| = |Y| = \Delta - 1$. Therefore v' cannot be v_1 or v_2 since $|S_{vv_1}| = 2$ and $|S_{vv_2}| = 2$ while $\Delta - 1 \geq 4$. Thus $v' \in N''(v)$ implying that $c'(u, u') \in F_v''$. □

Recalling that for configuration $B3$, $|F_v''| = 2$ and since $1 \in F_v''$, at least one of $3, 4$ belongs to F_v''. Without loss of generality let $3 \in F_v'$. Now recolor edge uu' using color 3 to get a coloring d from c'. The coloring d is valid by $Lemma$ 1 since $\{d(u, u'')\} \cap S_{uu'} = \{2\} \cap Y = \emptyset$. With respect to the coloring d we have $S_{uu'} = Y$ and $S_{uu''} \neq Y$. Thus by $subclaim$ 5.2, $d(u, u') \in F_v''$, a contradiction since $d(u, u') = 3 \notin F_v''$. Thus we have $Y = S_{uz}$ and $Y = S_{uw}$. □

Since $Y = S_{uz}$ and $Y = S_{uw}$, we can recolor edge uz and uw using color from F_v' (Recall that with respect to configuration $B3$, $|F_v'| = 2$) to get a new valid coloring c. The coloring c is valid by $Lemma$ 1 since $F_v' \cap S_{uz} = F_v' \cap Y = \emptyset$ and $F_v' \cap S_{uw} = F_v' \cap Y = \emptyset$. This reduces the situation to $F_u \subseteq F_v'$, a contradiction to $Claim$ 4.

subcase 1.2: v belongs to configuration $B4$.

We have $deg(v) = 6$ and $F_v'' = \{c'(v,a)\}$. Therefore in view of Claim 4, $c'(v,a)$ has to belong to F_u. Let $F_v = \{1,2,3,4,5\}$. Without loss of generality let $c'(u,w) = c'(v,v_1) = 2$ and $c'(u,z) = c'(v,a) = 1$. Now there are $\Delta - 2$ candidate colors for the edge vu. If none of them are valid then all these candidate colors are actively present in at least one of S_{uz} and S_{uw}. Let $X = C - \{1,2,3,4,5\}$.

Claim 6. $X \subseteq S_{uz}$.

Proof. Suppose not. Then let α be a color such that $\alpha \in X - S_{uz}$. This implies that α is actively present in S_{uw}. Hence there exists a $(2, \alpha, vu)$ critical path since $c'(u,w) = 2$. Now recolor edge uz using color α to get a coloring c''. The recoloring is valid since if there is a bichromatic cycle then it has to be a $(\alpha, 2)$ bichromatic cycle, implying that there existed a $(2, \alpha, uz)$ critical path in c', a contradiction in view of Fact 1 as there already existed a $(2, \alpha, vu)$ critical path. Now with respect to coloring c'', $F_v \cap F_u = \{2\}$ and therefore if none of the colors in $X - \{\alpha\}$ is valid for the edge vu, they all should be actively present in S_{uw}. Recalling that color α was actively present in S_{uw} we have all the colors of X actively present in S_{uw} and hence in S_{vv_1} implying that $|S_{vv_1}| \geq |X| = \Delta - 2 \geq 3$, a contradiction since $|S_{vv_1}| = 2$. Thus there exists a color valid for the edge vu, a contradiction to the assumption that G is a counter example. $\square$

Claim 7. $X \subseteq S_{uw}$.

Proof. Suppose not. Then let $X \nsubseteq S_{uw}$ and let α be a color such that $\alpha \in X - S_{uw}$. Recolor the edge uw using the color α. It is easy to see (by a similar argument used in the proof of Claim 6) that c'' is valid and all the colors of X are actively present in S_{uz} and hence in S_{va}.

Since $|X| = \Delta - 2$ and $|S_{va}| \leq \Delta - 1$, we have $|S_{va} - X| \leq 1$. If $S_{va} \neq X$, then the singleton set $S_{va} - X$ has to be a subset of $\{2,3,4,5\}$ since $1 \notin S_{va}$. Without loss of generality let $S_{va} - X = \{2\}$ (Reader may note that $\{2,3,4,5\} = F_v'$ and these four colors play symmetric roles in c'' and therefore we need to argue with respect to only one of them). Recall that $c''(v,v_1) = c'(v,v_1) = 2$ and $|S_{vv_1}| = 2$. Of the colors 3, 4 and 5 let $3 \notin S_{vv_1}$. Also let $c''(v,v_2) = 3$. Now delete the color on the edge vv_2 and recolor the edge va using color 3 to get a coloring d. We claim that the coloring d is valid: If $S_{va} = X$, then clearly it is valid by *Lemma* 1 since $S_{va} \cap S_{av} = \emptyset$. Otherwise we have $S_{va} - X = \{2\}$ and if there is a bichromatic cycle with respect to the coloring d, it has to be a $(2,3)$ bichromatic cycle. Since $d(v,v_1) = 2$, it means that $3 \in S_{vv_1}$, a contradiction to our assumption. Thus the coloring d is valid.

Now with respect to coloring d, we have $d(u,z) = 1$, $d(u,w) = \alpha$, $d(v,a) = 3$, $d(v,v_1) = 2$, $d(v,v_3) = 4$ and $d(v,v_4) = 5$. Edges vu and vv_2 are uncolored. Now let $X' = C - \{2,3,4,5\}$. Note that $|X'| \geq 5$ since $\Delta \geq 6$. We show below that there exists a color in X' that is valid for the edge vv_2:

- $S_{vv_2} \subset X'$. Now any color in $X' - S_{vv_2}$ is valid for the edge vv_2 by *Lemma* 1.

- $|S_{vv_2} \cap X'| = 1$. In this case exactly one color, say $\theta \in \{2,4,5\}$ is present in S_{vv_2} since $3 \notin S_{vv_2}$ (This is because $c'(v,v_2) = 3$). Now there are at least four candidate colors for the edge vv_2 since $|F_v \cup F_u| \leq 4 + 2 - 1 = 5$ and there are at least $\Delta + 3 \geq deg(v) + 3 = 6 + 3 = 9$ colors in C. If none of the candidate colors are valid then a (θ, γ) bichromatic cycle should form for each $\gamma \in X' - S_{vv_2}$. Since $\theta \in \{2,4,5\}$, we have $\theta = d(v,v_j)$ for $j = 1$, 3 or 4. It means that each of the (θ, γ) bichromatic cycle should contain the edge vv_j and thus $X' - S_{vv_2} \subseteq S_{vv_j}$. But $|X' - S_{vv_2}| \geq 5 - 2 + 1 \geq 4$ and $|S_{vv_j}| = 2$, a contradiction. Thus at least one color will be valid for the edge vv_2.

- $S_{vv_2} \cap X' = \emptyset$. Now all the colors in X' are candidates for the edge vv_2. If none of them are valid then all these candidate colors have to form bichromatic cycles with at least one of the colors in $S_{vv_2} \cap F_v$. Now since $c''(v,v_2) = 3$, color $3 \notin S_{vv_2}(d)$ and therefore 3 is not involved in any of these bichromatic cycles. Also since $|S_{vv_2}| = 2$, exactly two of the colors from $\{2,4,5\}$ and hence exactly two of the edges from $\{vv_1, vv_3, vv_4\}$ are involved in these bichromatic cycles. But we know that $|S_{vv_1}| = |S_{vv_3}| = |S_{vv_4}| = 2$. It follows that at most four bichromatic cycles can be formed. But $|X'| \geq 5$ and thus at least one color will be valid for the edge vv_2.

Let $\beta \in X'$ be a valid color for vv_2. Color the edge vv_2 using β to get a new coloring d'. Now:

- If $\beta \in C - \{1,2,3,4,5,\alpha\}$, then $F_v \cap F_u = \emptyset$ with respect to d', a contradiction to *Claim* 1.

- If $\beta \in \{1, \alpha\}$, then there are at least three candidate colors for the edge vu since $\Delta \geq 6$. Moreover we have $F_v \cap F_u = \{\beta\}$. If none of these three candidate colors are valid for the edge vu, then all of them have to be actively present in S_{vv_2}, implying that $|S_{vv_2}| \geq 3$, a contradiction since $|S_{vv_2}| = 2$. Therefore at least one of the three candidate colors is valid for the edge vu.

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Thus we have a valid color for edge vu, a contradiction to the assumption that G is a counter example. □

In view of $Claim\ 6$, $Claim\ 7$ and from $|S_{uz}|$, $|S_{uw}| \le \Delta - 1$ and $|X| = \Delta - 2$, it is easy to see that $|(S_{uz} \cup S_{uw}) - X| \le 2$. Thus recalling that $3, 4, 5 \notin X$, we infer that $\{3, 4, 5\} - (S_{uz} \cup S_{uw}) \ne \emptyset$. Now recolor the edge uz using a color $\mu \in \{3, 4, 5\} - (S_{uz} \cup S_{uw})$. Clearly μ is a candidate for the edge uz since $d'(u, w) = 2$ and $\mu \notin S_{uz}$. Moreover μ is valid for uz since if otherwise a $(2, \mu)$ bichromatic cycle has to be formed containing uw, implying that $\mu \in S_{uw}$, a contradiction. This reduces the situation to $F_u \subseteq F_v'$, a contradiction to $Claim\ 4$.

3.1.2   case 2: $|F_v \cap F_u| = 1$

Recall that by $Claim\ 3$ and $Claim\ 2$, v belongs to configurations $B3$, $B4$ or $B5$ and $deg(u) = 3$. As before $N(u) = \{v, w, z\}$. Also let $F_v \cap F_u = \{1\}$.

Claim 8. *With respect to any valid coloring of $G - \{vu\}$, $F_u \cap F_v' = \emptyset$. This implies that $F_v \cap F_u \subseteq F_v''$.*

Proof. Suppose not. Then without loss of generality let $c'(v, v_1) = c'(u, z) = 1$. Recalling $deg(u) = 3$, $|F_u| \le 2$ and thus $|F_u \cup F_v| \le (\Delta - 1) + 2 - 1 = \Delta$. It follows that there are at least three candidate colors for the edge vu. If none of the candidate colors are valid for the edge vu, then all these candidate colors have to be actively present in S_{vv_1}, implying that $|S_{vv_1}| \ge 3$, a contradiction since $|S_{vv_1}| = 2$. It follows that at least one of the three candidate colors is valid for the edge vu, a contradiction to the assumption that G is a counter example. □

In view of $Claim\ 8$, $F''(v) \ne \emptyset$ and therefore the vertex v cannot belong to configuration $B5$. We infer that v has to belong to either configuration $B3$ or $B4$. We take care of these two subcases separately below:

subcase 2.1: v belongs to configuration $B3$.

Since $deg(v) = 5$, we have $|F_v| = 4$. Let $F_v \cup F_u = \{1, 2, 3, 4, 5\}$. By Claim 8, we have $F_v \cap F_u = \{1\} \subseteq F_v'' = \{c'(v, a), c'(v, b)\}$. Without loss of generality let $c'(u, z) = c'(v, a) = 1$. Also let $c'(u, w) = 2$, $c'(v, b) = 3$, $c'(v, v_1) = 4$ and $c'(v, v_2) = 5$. Since $|F_v \cup F_u| = 5$, there are $\Delta - 2$ candidate colors for the edge vu. If none of them are valid then there exists a $(1, \alpha, vu)$ critical path for each $\alpha \in C - (F_v \cup F_u) = C - \{1, 2, 3, 4, 5\}$. Thus we have the following observation:

Observation 2. *With respect to the coloring c', each color in $C - \{1, 2, 3, 4, 5\}$ is actively present in S_{uz} as well as S_{va}.*

Claim 9. $S_{uz} = C - \{1, 3, 4, 5\}$ *and* $1, 4, 5 \in S_{uw}$.

Proof. Since $C - \{1, 2, 3, 4, 5\} \subseteq S_{uz}$ and $|S_{uz} - (C - \{1, 2, 3, 4, 5\})| \le 1$ we infer that at most one of $4, 5$ can be present in S_{uz}. Suppose one of $4, 5 \in S_{uz}$. Without loss of generality let $4 \in S_{uz}$. Now recolor edge uz using color 5. It is valid by $Lemma\ 1$ since $S_{uz} \cap S_{zu} = S_{uz} \cap \{2\} = \emptyset$. Thus we have reduced the situation to $F_u \cap F_v' \ne \emptyset$, a contradiction to $Claim\ 8$. Thus we have $4, 5 \notin S_{uz}$. Recolor edge uz using color 4 or 5. If any one of them is valid then we will have $F_u \cap F_v' \ne \emptyset$ with respect to this new coloring, a contradiction to $Claim\ 8$. It follows that none of them are valid. That is, bichromatic cycles get formed due to the recoloring. Clearly the bichromatic cycles have to be $(2, 4)$ and $(2, 5)$ bichromatic cycles since $c'(u, w) = 2$. Thus $2 \in S_{uz}$ and $4, 5 \in S_{uw}$. Recalling that $C - \{1, 2, 3, 4, 5\} \subseteq S_{uz}$ and $|S_{uz}| \le \Delta - 1$ we can infer that $S_{uz} = C - \{1, 3, 4, 5\}$.

Now if $1 \notin S_{uw}$, then assign color 1 to edge uw and the color 4 to edge uz. Clearly this recoloring is valid by $Lemma\ 1$ since $S_{zu} \cap S_{uz} = \{1\} \cap C - \{1, 3, 4, 5\} = \emptyset$. With respect to the new coloring, $F_u \cap F_v = \{1, 4\}$ which reduces the situation to $case\ 1$. Thus we infer that $1 \in S_{uw}$. Therefore we have $1, 4, 5 \in S_{uw}$. □

Claim 10. $|(C - \{1, 2, 3, 4, 5\}) - S_{uw}| \ge 2$.

Proof. Since $|S_{uw}| \le \Delta - 1$ there are at least four colors missing from S_{uw}. Thus even if colors 2 and 3 are missing from S_{uw} there should be at least two colors in $C - \{1, 2, 3, 4, 5\}$ that are absent in S_{uw} since $1, 4, 5 \in S_{uw}$ by $Claim\ 9$. □

Now discard the color on the edge uw to obtain a partial coloring d of G from c'.

Claim 11. *With respect to coloring d, $\forall \alpha \in C - \{1, 3, 4, 5\}$, there exists a $(1, \alpha, vu)$ critical path.*

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Proof. With respect to the coloring c', there existed $(1, \alpha, vu)$ critical path for all $\alpha \in C - (F_v \cup F_u) = C - \{1, 2, 3, 4, 5\}$ by *Observation* 2. These critical paths remain unaltered when we get d from c'. Thus these critical paths are present in d also. Thus it is enough to prove that there exists $(1, 2, vu)$ critical path with respect to the coloring d. Let $\theta \in (C - \{1, 2, 3, 4, 5\}) - S_{uw}$. Note that θ exists by *Claim* 10. Now color θ is a candidate for the edge uw since $\theta \notin S_{uw}$ and $d(u, z) = 1$. Recolor the edge uw using color θ to get a coloring d'. The coloring d' is valid since otherwise a $(1, \theta)$ bichromatic cycle has to be created due to the recoloring. This means that there existed a $(1, \theta, uw)$ critical path with respect to coloring c', a contradiction by *Fact* 1 as there already existed a $(1, \theta, vu)$ critical path with respect to the coloring c' by *Observation* 2. Thus the coloring d' is valid.

Now color 2 is a candidate for the edge vu. If it is valid we get a valid coloring for G. Thus it is not valid. This means that there exists a $(1, 2, vu)$ critical path with respect to the coloring d' since $F_v \cap F_u = \{1\}$ with respect to the coloring d'. Now it is easy to see that this $(1, 2, vu)$ critical path will also exist with respect to coloring d. Thus with respect to the coloring d, $\forall \alpha \in C - \{1, 3, 4, 5\}$, there exists a $(1, \alpha, vu)$ critical path. $\qquad \square$

Observation 3. *Let $Q = (C - \{1, 3, 4, 5\}) - S_{uw}$. From Claim 10, we know that $|(C - \{1, 2, 3, 4, 5\}) - S_{uw}| \geq 2$. Since $c'(u, w) = 2$ we have $2 \notin S_{uw}$. From this we can infer that $2 \in Q$. Thus $|Q| \geq 3$.*

Claim 12. *There exists a color $\gamma \in Q$ such that γ is valid for the edge vv_1 or vv_2.*

Proof. Recall that $|S_{vv_1}| = 2$, $|S_{vv_2}| = 2$ and by *Observation* 2, $|Q| \geq 3$.

- If $S_{vv_1} \subset Q$ or $S_{vv_2} \subset Q$. Without loss of generality let $S_{vv_1} \subset Q$. Let γ be a color in $Q - S_{vv_1}$. Recolor edge vv_1 using color γ to get a coloring d'. The coloring d' is valid by *Lemma* 1 as $S_{vv_1} \cap S_{v_1v} = \emptyset$ since $Q \cap F_v = \emptyset$.

- If $S_{vv_1} \nsubseteq Q$ and $S_{vv_2} \nsubseteq Q$. In this case, at most one color in Q can be in S_{vv_1} and the same holds true for S_{vv_2}. Thus all the colors of Q except for one are candidates for edge vv_1 and all the colors of Q except for one are candidates for edge vv_2. Since $|Q| \geq 3$, we can infer that there exists a color $\gamma \in Q$ which is a candidate for both vv_1 and vv_2.

  subclaim *Color γ is valid either for the edge vv_1 or for the edge vv_2.*

  Proof. Recolor vv_1 using color γ. If γ is valid, we are done. If it is not valid, then there has to be a (γ, θ) bichromatic cycle getting formed, where $\theta \in F_v - \{d(v, v_1)\} = F_v - \{4\} = \{1, 3, 5\}$. But this cannot be a $(\gamma, 5)$ bichromatic cycle since $\gamma \notin S_{vv_2}$ (recall that $d(v, v_2) = c'(v, v_2) = 5$). Also this cannot be a $(\gamma, 1)$ bichromatic cycle since otherwise it implies that there exists a $(1, \gamma, vv_1)$ critical path with respect to the coloring d, a contradiction in view of *Fact* 1 as there already exists a $(1, \gamma, vu)$ critical path by *Claim* 11. Thus it has to be a $(3, \gamma)$ bichromatic cycle, implying that there existed a $(3, \gamma, vv_1)$ critical path with respect to the coloring d.

  If γ is not valid for the edge vv_1 we recolor edge vv_2 instead, using color γ to get a coloring d' form d. We claim that the coloring d' is valid. This is because there cannot be a $(\gamma, 4)$ bichromatic cycle since $\gamma \notin S_{vv_1}$ (recall that $d(v, v_1) = c'(v, v_1) = 4$). Also there cannot be a $(\gamma, 1)$ bichromatic cycle since otherwise it implies that there exists a $(1, \gamma, vv_2)$ critical path with respect to the coloring d, a contradiction in view of *Fact* 1 as there already exists a $(1, \gamma, vu)$ critical path by *Claim* 11. Finally there cannot be a $(3, \gamma)$ bichromatic cycle because this implies that there existed a $(3, \gamma, vv_2)$ critical path with respect to the coloring d, a contradiction by *Fact* 1 since there already existed a $(3, \gamma, vv_1)$ critical path with respect to the coloring d. Thus the coloring d' is valid. $\qquad \square$

$\qquad \square$

In view of Claim 12, without loss of generality let $\gamma \in Q$ be valid for the edge vv_1. Now we recolor the edge vv_1 using color γ to get a coloring d'.

We claim that none of the colors in S_{uw} were altered in this recoloring. This is because if they are altered then vv_1 has to be an edge incident on w and thus one of the end points of vv_1 has to be w. Since v cannot be w, either v_1 should be w. But we know that $deg(v_1) = 3$. Recall that $1, 4, 5 \in S_{uw}$ and thus $deg(w) \geq 4$. Thus v_1 cannot be w. Thus none of the colors of S_{uw} are modified while getting d' from d. We infer that $\gamma \notin S_{uw}$ since $Q \cap S_{uw} = \emptyset$. Therefore γ is a candidate for the edge uw since $d'(u, z) = 1$. Now color the edge uw using the color γ to get a coloring d''. If the coloring d'' is valid, then we have $F_u \cap F_v = \{1, \gamma\}$. This reduces the situation to *case* 1.

On the other hand if the coloring d'' is not valid then there has to be a bichromatic cycle formed due to the recoloring of edge uw. Since $d''(u,z) = 1$, it has to be a $(1,\gamma)$ bichromatic cycle. Recall that there existed a $(1,\gamma,vu)$ critical path with respect to the coloring d. Note that to get d'' from d we have only recolored two edges namely vv_1 and uw, both with color γ. Clearly these recolorings cannot break the $(1,\gamma,vu)$ critical path that existed in d, but only can extend it. Thus we can infer that in d'' the $(1,\gamma)$ bichromatic cycle passes through v and hence through the edges va and vv_1. Now recolor edge va using color 4 to get a coloring c. Recall that $S_{va} = C - \{1,3,4,5\}$ by Claim 11 and $S_{av} = F_v - \{c''(v,a)\} = \{3,5,\gamma\}$. Therefore color 4 is indeed a candidate for edge va. Note that by recoloring va using color 4, we have broken the $(1,\gamma)$ bichromatic cycle that existed in d''. Now we claim that the coloring c is valid. Note that $S_{va} \cap S_{av} = S_{va} \cap \{3,5,\gamma\} = \{\gamma\}$. If a bichromatic cycle gets formed due to this recoloring then it has to be $(4,\gamma)$ bichromatic cycle, implying that $4 \in S_{vv_1}$. But $S_{vv_1}(c) = S_{vv_1}(d'') = S_{vv_1}(d)$ and $4 \notin S_{vv_1}(d)$ since $d(v,v_1) = 4$. Thus $4 \notin S_{vv_1}(c)$, a contradiction. Thus the coloring c is valid. With respect to the coloring c, we have $F_v \cap F_u = \{\gamma\} \subset F'_v$, a contradiction to $Claim\ 8$.

subcase 2.2: v belongs to configuration $B4$.

We have $deg(v) = 6$ and therefore $|F_v| = 5$. Moreover $|F''_v| = 1$ and $|F'_v| = 4$. By $Claim\ 8$, $F_v \cap F_u = \{1\} \subseteq F''_v$. Without loss of generality let $c'(u,z) = c'(v,a) = 1$. Also let $c(u,w) = 2$, $F'_v = \{3,4,5,6\}$ and $Z = \{3,4,5,6\}$. There are $\Delta - 3$ candidate colors for the edge vu. If none of them are valid then there exist $(1,\alpha,vu)$ critical path for each $\alpha \in C - (F_v \cup F_u) = C - \{1,2,3,4,5,6\}$. Thus we have the following observation:

Observation 4. *With respect to the coloring c', each color in $C - \{1,2,3,4,5,6\}$ is actively present in S_{uz} as well as S_{va}.*

Claim 13. $S_{uz} \supseteq C - \{1,3,4,5,6\}$ *and* $1 \in S_{uw}$. *Also at least three of the colors from Z are present in S_{uw}.*

Proof. As we have seen above $C - \{1,2,3,4,5,6\} \subseteq S_{uz}$. Suppose $2 \notin S_{uz}$. Note that every color in $C - (S_{uz} \cup S_{zu})$ is a candidate for uz. Now $S_{zu} = \{c'(u,w)\} = \{2\}$. Moreover $|S_{uz}| \leq \Delta - 1$ and thus S_{uz} can have at most two more colors other than those in $C - \{1,2,3,4,5,6\}$. From this we can infer that at least two of the colors in Z are candidates for the edge uz. They are also valid by $Lemma\ 1$ since $S_{uz} \cap S_{zu} = S_{uz} \cap \{2\} = \emptyset$. Thus we can reduce the situation to $F_u \cap F'_v \neq \emptyset$, by assigning one of the valid colors from Z to uz, thereby getting a contradiction to $Claim\ 8$. Thus we infer that $2 \in S_{uz}$. Therefore we get $S_{uz} \supseteq C - \{1,3,4,5,6\}$. Since $|S_{uz}| \leq \Delta - 1$ and $|C - \{1,3,4,5,6\}| = \Delta - 2$ we can infer that $|Z \cap S_{uz}| \leq 1$.

If any one of the colors in $Z - S_{uz}$ is valid for the edge uz, then it will reduce the situation to $F_u \cap F'_v \neq \emptyset$, a contradiction to $Claim\ 8$. Thus none of these colors are valid for the edge uz. Therefore there should be bichromatic cycles getting formed when we try to recolor edge uz using any of these colors. These bichromatic cycles have to be $(2,\mu)$ bichromatic cycles for each color $\mu \in Z - S_{uz}$ since $c'(u,w) = 2$. Thus we can infer that at least three of the colors from Z are present in S_{uw} since $|Z - S_{uz}| \geq 4 - 1 = 3$.

Now if $1 \notin S_{uw}$, then assign color 1 to edge uw and a color $\mu \in Z - S_{uz}$ to edge uz. Clearly this recoloring is valid by $Lemma\ 1$ since $S_{zu} \cap S_{uz} = \{1\} \cap S_{uz} = \emptyset$ ($1 \notin S_{uz}$ since $c'(u,z) = 1$). With respect to the new coloring, $F_u \cap F_v = \{1,\mu\}$ which reduces the situation to $case\ 1$. Thus we infer that $1 \in S_{uw}$. $\square$

Claim 14. $|(C - \{1,2,3,4,5,6\}) - S_{uw}| \geq 2$.

Proof. Since $|S_{uw}| \leq \Delta - 1$, we have $|C - S_{uw}| \geq 4$. Now since $|Z \cap S_{uw}| \geq 3$ and $1 \in S_{uw}$, $|\{1,2,3,4,5,6\} \cap S_{uw}| \geq 4$. It follows that $|(C - S_{uw}) \cap \{1,2,3,4,5,6\}| \leq 2$ and the Claim follows. $\square$

Now discard the color on the edge uw to obtain a partial coloring d of G from c'.

Claim 15. *With respect to coloring d, $\forall \alpha \in C - \{1,3,4,5,6\}$, there exists a $(1,\alpha,vu)$ critical path and thus $C - \{1,3,4,5,6\} \subseteq S_{va}$.*

Proof. With respect to the coloring c', there existed a $(1,\alpha,vu)$ critical path for each $\alpha \in C - (F_v \cup F_u) = C - \{1,2,3,4,5,6\}$ by $Observation\ 4$. These critical paths remain unaltered when we get d from c'. Thus these critical paths are present in d also. Thus it is enough to prove that there exists a $(1,2,vu)$ critical path with respect to the coloring d. Let $\theta \in (C - \{1,2,3,4,5,6\}) - S_{uw}$. Note that θ exists by $Claim\ 14$. Now color θ is a candidate for the edge uw since $\theta \notin S_{uw}$ and $d(u,z) = 1$. Recolor the edge uw using color θ to get a coloring d'. The coloring d' is valid since otherwise a $(1,\theta)$ bichromatic cycle has to be created due to the recoloring. This means that there existed a $(1,\theta,uw)$ critical path

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with respect to coloring c', a contradiction by $Fact$ 1 as there already existed a $(1, \theta, vu)$ critical path with respect to the coloring c' by $Observation$ 4. Thus the coloring d' is valid.

Now color 2 is a candidate for the edge vu. If it is valid we get a valid coloring for G. Thus it is not valid. This means that there exists a $(1, 2, vu)$ critical path with respect to the coloring d' since $F_v \cap F_u = \{1\}$ with respect to the coloring d'. Now it is easy to see that this $(1, 2, vu)$ critical path will also exist with respect to coloring d. Thus with respect to the coloring d, $\forall \alpha \in C - \{1, 3, 4, 5, 6\}$, there exists a $(1, \alpha, vu)$ critical path. $\qquad \square$

Observation 5. *Let* $Q = (C - \{1, 3, 4, 5, 6\}) - S_{uw}$. *From Claim 14, we know that* $|(C - \{1, 2, 3, 4, 5, 6\}) - S_{uw}| \geq 2$. *Since* $c'(u, w) = 2$ *we have* $2 \notin S_{uw}$. *From this we can infer that* $2 \in Q$. *Thus* $|Q| \geq 3$.

Recall that $|S_{vv_i}| = 2$, for $i \in \{1, 2, 3, 4\}$ and by $Observation$ 5, $|Q| \geq 3$. We know that $S_{va} \supseteq C - \{1, 3, 4, 5, 6\}$ by $Claim$ 15. Since $|C - \{1, 3, 4, 5, 6\}| = \Delta - 2$ and $|S_{va}| \leq \Delta - 1$ we have $|Z \cap S_{va}| = |\{3, 4, 5, 6\} \cap S_{va}| \leq 1$. We make the following assumption:

Assumption 2. *If* $Z \cap S_{va} \neq \emptyset$, *let* $\{\alpha\} = Z \cap S_{va}$ *and let* $d(v, v_t) = \alpha$, *where* $t \in \{1, 2, 3, 4\}$. *Let* $\beta \in (Z - \{\alpha\}) - S_{vv_t}$. *If* $Z \cap S_{va} = \emptyset$, *then let* β *be any color in* Z.

We now plan to recolor one of the edges in $\{vv_1, vv_2, vv_3, vv_4\}$ using a specially selected color $\gamma \in Q$. After this we will also use the same color γ to recolor edge uw, with the intention of reducing the situation to $case$ 1. Below we give the recoloring procedure for the rest of the proof starting from the current coloring d in 3 steps. The final coloring c of $G - \{vu\}$ that we obtain at the end of $Step3$ will give the required contradiction.

Step1: With respect to the coloring d,

(i) **If one of the edges vv_i, for $i \in \{1, 2, 3, 4\}$ is such that $S_{vv_i} \subset Q$, then recolor that edge with any color $\gamma \in Q - S_{vv_i}$. We call the edge that we chose to recolor as $(v, v_{t'})$.**

(ii) **If $\forall i \in \{1, 2, 3, 4\}$, $S_{vv_i} \not\subseteq Q$, then we select an edge $vv_{t'}$, where $t' \in \{1, 2, 3, 4\}$ such that $d(v, v_{t'}) = \beta$ (See $Assumption$ 2). Now recolor the edge $vv_{t'}$ with a suitably selected (see the proof of $Claim$ 16) color in $Q - S_{vv_{t'}}$.**

The resulting coloring after performing $Step1$ is named d'.

Claim 16. *There exists a color* $\gamma \in Q$ *such that the coloring* d' *obtained after* $Step1$ *is valid.*

Proof. At the beginning of $Step1$, we had the following possible cases:

(i) **One of the edges vv_i, for $i \in \{1, 2, 3, 4\}$ is such that $S_{vv_i} \subset Q$:**
Let γ be a color in $Q - S_{vv_i}$. Recolor edge vv_i using color γ to get a coloring d'. The coloring d' is valid by $Lemma$ 1 as $S_{vv_i} \cap S_{v_i v} = \emptyset$ since $Q \cap F_v = \emptyset$.

(ii) $S_{vv_i} \not\subseteq Q$, **for each $i \in \{1, 2, 3, 4\}$:**
Let t' be as defined in $Step1$. Clearly all the colors in $Q - S_{vv_{t'}}$ are candidates for $vv_{t'}$ since $Q \cap F_v = \emptyset$. Note that since $S_{vv_i} \not\subseteq Q$ we have $|Q \cap S_{vv_{t'}}| \leq 1$ and therefore $|Q - S_{vv_{t'}}| \geq 2$. If any one of the candidate colors is valid for the edge $vv_{t'}$, the statement of the Claim is obviously true. On the other hand if none of them are valid, then there has to be a (γ, θ) bichromatic cycle getting formed, for some $\theta \in F_v - \{d(v, v_{t'})\} = F_v - \{\beta\}$ when we try to recolor edge $vv_{t'}$ using color γ, for each $\gamma \in Q - S_{vv_{t'}}$. Note that $\theta \neq 1$ because if a $(\gamma, 1)$ bichromatic cycle gets formed, then there has to be a $(1, \gamma, vv_{t'})$ critical path with respect to the coloring d, a contradiction in view of $Fact$ 1 as there already exists a $(1, \gamma, vu)$ critical path by $Claim$ 15. Thus $\theta \in F'_v - \{d(v, v_{t'})\}$ since $F''_v = \{1\}$. Therefore we have $|(F'_v - \{d(v, v_{t'})\}) \cap S_{vv_{t'}}| \geq 1$. We have the following cases:

– $|(F'_v - \{d(v, v_{t'})\}) \cap S_{vv_{t'}}| = 1$: Let $S_{vv_{t'}} \cap (F'_v - \{d(v, v_{t'})\}) = d(v, v')$, for $v' \in \{v_1, v_2, v_3, v_4\} - \{v_{t'}\}$. Thus all the candidate colors of $vv_{t'}$, namely all the colors of $Q - S_{vv_{t'}}$ should form bichromatic cycles passing through the edge vv', implying that $Q - S_{vv_{t'}} \subset S_{vv'}$. But $|Q - S_{vv_{t'}}| \geq 2$ and $|S_{vv'}| = 2$. Thus $S_{vv'} = Q - S_{vv_{t'}} \subseteq Q$, a contradiction.

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- $|(F'_v - \{d(v, v_{t'})\}) \cap S_{vv_{t'}}| = 2$: This means that $S_{vv_{t'}} \subseteq F'_v$ and therefore we have $Q \cap S_{vv_{t'}} = \emptyset$. Thus $|Q - S_{vv_{t'}}| = |Q| \geq 3$. Therefore there are at least three candidate colors for the edge $vv_{t'}$. Let $S_{vv_{t'}} \cap (F'_v - \{d(v, v_{t'})\}) = \{d(v, v'), d(v, v'')\}$, for $v', v'' \in \{v_1, v_2, v_3, v_4\} - \{v_{t'}\}$. Since for each candidate color we have a bichromatic cycle, we can infer that there are at least three bichromatic cycles, each of them passing through either vv' or vv''. Thus at least two bichromatic cycles have to pass through one of vv' and vv''. But since $|S_{vv'}| = 2$ and $|S_{vv''}| = 2$, we can infer that either $S_{vv'} \subseteq Q$ or $S_{vv''} \subseteq Q$, a contradiction.

$\square$

Step2: Let γ be the color which was used to recolor the edge $vv_{t'}$ in $Step1$. Now recolor edge uw with color γ to get a coloring d''.

Claim 17. *The coloring d'' is proper.*

Proof. We claim that none of the colors in S_{uw} were altered in $Step1$. This is because if they are altered then the edge $vv_{t'}$ should be incident on w and thus one of the end points of $vv_{t'}$, where $t' \in \{1, 2, 3, 4\}$, has to be w. Since v cannot be w, $v_{t'}$ should be w. But we know that $deg(v_i) = 3$. Recall that $|Z \cap S_{uw}| \geq 3$ by $Claim$ 13 and thus $|S_{uw}| \geq 3$. Therefore $deg(w) \geq 4$. Thus $v_{t'}$ cannot be w. Thus none of the colors of S_{uw} are modified while getting d' from d. Recall that $Q = (C - \{1, 3, 4, 5, 6\}) - S_{uw}$ and thus $\gamma \notin S_{uw}$. Therefore γ is a candidate for the edge uw since $d(u, z) = 1$. Thus the coloring d'' is proper. $\square$

If the coloring d'' is valid, then we have $F_u \cap F_v = \{1, \gamma\}$ for a valid coloring of $G - \{vu\}$. This reduces the situation to $case$ 1. Thus coloring d'' is not valid. Since the coloring d'' is not valid, there has to be a bichromatic cycle formed due to the recoloring of edge uw. Since $d''(u, z) = 1$, it has to be a $(1, \gamma)$ bichromatic cycle. Recall that there existed a $(1, \gamma, vu)$ critical path with respect to the coloring d by $Claim$ 15. Note that to get d'' from d we have only recolored two edges namely $vv_{t'}$ and uw, both with color γ. Clearly these recolorings cannot break the $(1, \gamma, vu)$ critical path that existed in d, but can only extend it. Thus we can infer that in d'' the $(1, \gamma)$ bichromatic cycle passes through v and hence through the edges va and $vv_{t'}$. Also note that this can happen only when we have $1 \in S_{vv_{t'}}$. Thus $S_{vv_{t'}} \nsubseteq Q$. It means that substep (ii) of $Step1$ was executed; and the color on $vv_{t'}$ with respect to coloring d was β (from $Assumption$ 2). We break the $(1, \gamma)$ bichromatic cycle as follows:

Step3: Recolor the edge va with color β (see in $Assumption$ 2) to get a coloring c.

Claim 18. *The coloring c is valid.*

Proof. Recall by $Assumption$ 2 that $\beta \notin S_{va}$. Also clearly $\beta \notin F_v(d'')$ since we recolored $vv_{t'}$ by a color $\gamma \in Q$ to get d'' form d ($\beta \neq \gamma$ since $\beta \in F_v(d)$ and $F_v(d) \cap Q = \emptyset$). Therefore color β is a candidate for edge va. Note that by recoloring va using color β, we have broken the $(1, \gamma)$ bichromatic cycle that existed in d''. We claim that the coloring c is valid. Otherwise there has to be a bichromatic cycle involving β and a color in $S_{va} \cap S_{av}$. But $S_{av} = (Z - \{\beta\}) \cup \{\gamma\} = (\{3, 4, 5, 6\} - \{\beta\}) \cup \{\gamma\}$. Since with respect to d'' there was a $(1, \gamma)$ bichromatic cycle passing through the edges va and $d''(v, a) = 1$, we have $\gamma \in S_{va} \cap S_{av}$. But there cannot be a (β, γ) bichromatic cycle getting formed in c since such a cycle should contain edge $vv_{t'}$ and thus $\beta \in S_{vv_{t'}}$. But $S_{vv_{t'}}(c) = S_{vv_{t'}}(d'') = S_{vv_{t'}}(d)$ and $\beta \notin S_{vv_{t'}}(d)$ since $d(v, v_{t'}) = \beta$. Thus $\beta \notin S_{vv_1}(c)$, a contradiction. Thus there cannot be a (β, γ) bichromatic cycle.

Thus if the coloring c is not valid then there has to be a bichromatic cycle involving β and one of the colors in $Z - \{\beta\} \cap S_{va}$. We know by $Assumption$ 2 that $Z \cap S_{va} = \alpha$. Thus it has to be a (β, α) bichromatic cycle. Since $c(v, v_t) = d(v, v_t) = \alpha$, this bichromatic cycle contains the edge vv_t and hence $\beta \in S_{vv_t}$, a contradiction to the way β was selected in $Assumption$ 2. Thus there cannot be a (β, α) bichromatic cycle. Thus the coloring c is valid. $\square$

With respect to the coloring c, we have $F_v \cap F_u = \{\gamma\} \subset F'_v$, a contradiction to $Claim$ 8.

3.2 There exists no vertex v that belongs to one of the configurations $B2$, $B3$, $B4$ or $B5$

This means that there exists a vertex v that belongs to configuration $B1$, i.e., $deg(v) = 2$. Let $Q = \{u \in V : deg(u) = 2\}$. First we claim that Q is an independent set in G. Otherwise let $u', u \in Q$ be such that $(u, u') \in E(G)$. Now since G is a minimum counter example, $G - \{uu'\}$ is acyclically edge colorable using $\Delta + 3$ colors. Let c' be a valid coloring of $G - \{uu'\}$. Now if $F_u \cap F_{u'} = \emptyset$, then there are $\Delta + 3 - 2 = \Delta + 1$, candidate colors for the edge uu'. Since $S_{uu'} \cap S_{u'u} = \emptyset$, by $Lemma$ 1, all the candidate colors are valid for the edge uu'. On the other hand if $|F_u \cap F_{u'}| = 1$, then there are $\Delta + 3 - 1 = \Delta + 2$ candidate colors for the edge uu'. Let $N(u) = \{u', u''\}$. If none of them are valid then all those candidate colors have to be actively present in $S_{uu''}$, implying that $|S_{uu''}| \geq \Delta + 2$, a contradiction since $|S_{uu''}| \leq \Delta - 1$. Thus there exists a valid coloring of G, a contradiction to the assumption that G is a counter example. We infer that Q is an independent set in G.

Now delete all the vertices in Q from G to get a graph G'. Clearly the graph G' has at most $2|V(G')| - 1$ edges since Q is an independent set. It follows by $Lemma$ 4 that there should be a vertex v' in G' such that v' is the pivot of one of the configurations $B1 - B5$, say $B' = \{v'\} \cup N_{G'}(v')$. But with respect to graph G, $\{v'\} \cup N_{G'}(v')$ did not form any of the configurations $B1 - B5$. This means that the degree of at least one of the vertices in $\{v'\} \cup N_{G'}(v')$ should have got decreased by the removal of Q from G. Let P be the set of vertices in $\{v'\} \cup N_{G'}(v')$ whose degrees got reduced due to the removal of Q from G, i.e., $P = \{z \in \{v'\} \cup N_{G'}(v') : deg_{G'}(z) < deg_G(z)\}$.

For a vertex $x \in V(G)$, let $M''_G(x) = \{u \in N_G(x) : deg_G(u) > 3\}$ and $M'_G(x) = N_G(x) - M''_G(x)$. Note that in all the configurations defined in $Lemma$ 4, the main criteria which characterizes each configuration is the degree of the pivot v' and the degrees of the vertices in $N'(v')$. We make the following claim:

Claim 19. *There exists a vertex x in P such that $|M''_G(x)| \leq 3$.*

Proof. It is easy to see that $M''_G(x) \subseteq N_{G'}(x)$. If there exists a vertex in P, whose degree is at most 3, say x, then we have $|M''_G(x)| \leq 3$. Thus we can assume that the degree of any vertex in P is at least 4.

Now suppose the pivot vertex v' is in P. Then let $x = v'$. It is clear that v' has to be in one of the configuration $B3 - B5$. In any of these configurations there can be at most two neighbours with degree greater than 3. Note that in this case all the degree 3 neighbours of $x = v'$ in G' are of degree 3 in G also since otherwise P will contain a vertex of degree at most 3, a contradiction. Thus we have $|M''_G(x)| \leq 2$.

The only remaining case is when $v' \notin P$. Since the degree of v' has not changed and $\{v'\} \cup N_G(v')$ was not in any configuration in G, it means that one of the vertex in $N'(v')$ has had its degree decreased. We call that vertex as x. Since the degree of any vertex in P is at least 4, $deg_{G'}(x) \geq 4$. Since we can have degree ≥ 4 vertex in $N'(v')$ only if $\{v'\} \cup N_G(v')$ forms a configuration $B2$, we infer that $deg_{G'}(x) = 4$. Moreover $deg_{G'}(v') = deg_G(v') = 3$. Thus we have $|M''_G(x)| \leq |N_{G'}(x) - \{v'\}| \leq 4 - 1 = 3$.

Thus we have $|M''_G(x)| \leq 3$. $\qquad\blacksquare$

In G, let y be a two degree neighbour of vertex x - selected in $Claim$ 19 - such that $N(y) = \{x, y'\}$. Now by induction $G - \{xy\}$ is acyclically edge colorable using $\Delta + 3$ colors. Let c' be a valid coloring of $G - \{xy\}$. With respect to the coloring c' let $F'_x(c') = \{c'(x, z) | z \in M'(x)\}$ and $F''_x(c') = \{c'(x, z) | z \in M''(x)\}$ i.e., $F''_x = F_x - F'_x$.

Now if $c'(y, y') \notin F_x$ we are done as there are at least three candidate colors which are also valid by $Lemma$ 1. We know by $Claim$ 19 that $|F''_x| \leq 3$. If $c'(y, y') \in F'_x$, then let $c = c'$. Else if $c'(y, y') \in F''_x$, then recolor edge yy' using a color from $C - (S_{yy'} \cup F''_x)$ to get a coloring c (Note that $|C - (S_{yy'} \cup F''_x)| \geq \Delta + 3 - (\Delta - 1 + 3) = 1$ and since y is a pendant vertex in $G - \{xy\}$ the recoloring is valid). Now if $c(y, y') \notin F_x$ the proof is already discussed. Thus $c(y, y') \in F'_x$.

With respect to coloring c, let $a \in M'(x)$ be such that $c(x, a) = c(y, y') = 1$. Now if none of the candidate colors in $C - (F_x \cup F_y)$ are valid for the edge xy, then all those candidate colors have to be actively present in S_{xa}, implying that $|S_{xa}| \geq |C - (F_x \cup F_y)| \geq \Delta + 3 - (\Delta - 1 + 1 - 1) = 4$, a contradiction since $|S_{xa}| \leq 2$ (Recall that $a \in M'(x)$ and $deg(a) \leq 3$). Thus we have a valid color for the edge xy, a contradiction to the assumption that G is a counter example. $\qquad\blacksquare$

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